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Benchmarking Time-Optimal Control Inputs for Flexible Systems

Michael C. Reynolds and Peter H. Meckl
Purdue University, West Lafayette, Indiana 47907

Command shaping techniques adjust the motion commands to flexible systems so that residual vibration is reduced. However, with so many command shaping methods available, benchmarking is necessary so that the contributions and performance of a particular input can be assessed. A closed-form solution for the move plus settling time of a rigid-body input when applied to a flexible system with damping is developed. The move plus settling time of the rigid-body solution creates an upper bound on most feasible solutions for systems with actuator constraints. A lower bound on the point-to-point motion is the pure move time of the rigid-body input. The usefulness and importance of benchmarking using the rigid-body solution, as well as other inputs, is described. Results emphasize the importance of command shaping when the system damping ratio is less than 0.1.

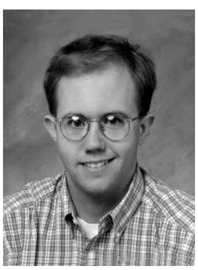
I. Introduction

In an increasingly automated and mechanized world, mechanical systems must be able to perform fast and accurate motions. A variety of applications exist that demand fast motions with minimum residual vibration. These applications include but are not limited to robotics,[1] spacecraft,[2,3] disk drives,[4] computer chip manufacturing, sewing machines, and telescopes. With so many applications that would benefit from faster motion, there has been much research into this field.

Simple methods to create faster motions in flexible systems include adding rigidity or stiffness to the system. Often these techniques are unfeasible due to cost or physical limitations. Thus, faster motions usually increase residual vibration, which can lead to longer move times because the system must settle before the move is completed. With the proper control scheme, fast motions can be produced with minimal residual vibration and without the drawbacks of altering the physical system.

Traditional methods of control involve using a feedback system to achieve the desired performance specifications. Although much research has gone into using feedback control in flexible systems, the results are rarely time optimal.

Time-optimal control is part of a broader area known as command shaping. Much active research is currently occurring in the field of command shaping. One subset of command shaping is waveform synthesis, which is the design of smooth motion profiles. One waveform synthesis method comes from inverse dynamics.[5] When the system model is inverted, an input can be found by specifying the output. Unfortunately, the selected output trajectory does not always lead to an input, and it can be difficult to find the optimal trajectory. Bhat and Miu[6] have shown how arbitrary control waveforms can be optimized using finite-time Laplace transforms. Using ramped sinusoid forcing functions, Meckl and Kinceler[4] have been able to achieve fast motions with minimal residual vibration. Because the coefficients of the ramped sinusoid can be designed to approximate a rectangular forcing function, the move times are closer to the time-optimal bang–bang input than most other command shaping strategies. Another advantage of the ramped sinusoid is that the amount of insensitivity to parameter uncertainty can easily be built into the design. S-curve velocity profiles have shown great potential in improving the performance of many systems.[7] S-curves are produced when the force is gradually applied and removed instead of being switched between maximum positive and negative values. Meckl and Arestides[8] have optimized the selection of S-curve profiles such that there are significant reductions in residual vibration when compared to a trapezoidal velocity profile. Although all of these methods have advantages, they all produce outputs that are suboptimal with respect to move time.



Michael Reynolds received his B.S. in Mechanical Engineering from Marquette University in 1996 and his M.S. in Mechanical Engineering from Purdue University in 1999. Michael is currently working on his Ph.D. in Mechanical Engineering at Purdue. His research focuses on developing and benchmarking time-optimal commands for flexible systems; reynoldm@ecn.purdue.edu.

Peter H. Meckl obtained the Ph.D. in Mechanical Engineering from MIT in 1988. He joined the faculty in the School of Mechanical Engineering at Purdue University in 1988, where he has been an Associate Professor since 1994. Dr. Meckl's research interests are primarily in dynamics and control of machines, with emphasis on vibration reduction and motion control. His teaching responsibilities include undergraduate courses in systems modeling, measurement systems, and control, and graduate courses in advanced control design and microprocessor control. Dr. Meckl was selected as an NEC Faculty Fellow from 1990 to 1992. He received the Ruth and Joel Spira Award for outstanding teaching in 2000; meckl@ecn.purdue.edu.

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Another command shaping technique referred to as optimal open-loop profiling solves a two-point boundary-value problem through the minimization of a performance index. Farrenkopf[8] uses an optimal open-loop profile to show how the product of the lowest natural frequency and the maneuver time is an important parameter in determining the type of optimal response. The same parameter is used in this work so that many inputs can be compared without reference to dimensions. The biggest drawbacks of optimal open-loop profiling are that it can be difficult to compute and that the inputs are normally sensitive to parameter uncertainty.

A third category of command shaping designs is input shaping. Input shaping is the convolution of an unshaped input with a sequence of impulses. These impulses act like a filter, removing energy near the system's natural frequency. Although generally the input shaper is placed before the feedback loop, it can also be designed inside the loop.[9] The major difference between traditional filtering and input shaping is that the design of the input shaper is based on decaying sinusoids in the time domain rather than the frequency domain.[10] Input shapers have the advantage of being effective with both open-loop and closed-loop systems because the input shaper comes before the feedback loop. Input shapers can be quickly calculated and used with arbitrary inputs in real time. Many different input shaping designs exist. Some designs use only positive impulses, whereas others use positive and negative impulses.[11] Negative shapers create faster move times than shapers with only positive impulses, but they can cause excitation of higher modes and saturation of actuators. Other input shapers more explicitly account for robustness concerns. Pao[12] describes several types of input shapers to account for uncertainty in natural frequency and damping. The zero-vibration, zero-derivative shaper causes the residual vibration and the derivative of the residual vibration to be zero at the nominal frequency. This gives a wider range of frequencies that have minimal residual vibration. All of these designs that add insensitivity have the drawback of longer move times when compared to the time-optimal solution. Pao and Singhose[13] show that time-optimal forcing functions can be generated using a special case of input shaping.

With all of the command shaping methods available to the controls community, a benchmark is necessary to compare different design strategies, test the efficiency of a particular input, and serve as the best input in certain cases.[14]

The main benchmark developed in this paper applies to the point-to-point motion of a rigid body. The rigid-body solution is referred to as the single switch (SS) solution because its input has only one internal switch. The SS solution is a good benchmark for several reasons. First, treating the system as a rigid body results in the fastest possible move time. The SS solution is analogous to a Carnot efficiency limit in thermodynamics; no input will have a shorter move time with the same forcing constraints. Second, the move plus settling time of an SS input can be found in a closed-form equation. This allows for a quick comparison for many different systems. Third, because the SS input ignores the flexibility and damping of the real system, it is infinitely robust to flexible parameter variation. Whereas parameter variation and uncertainty do affect the output of a system with an SS input, the average effect of parameter variation on the output is minimal when compared to most command shaping methods.

The following section will describe the system and the dimensionless parameters used in this paper. Next, the closed-form solution for the move plus settling time of the SS benchmark will be derived. Three other inputs that are time optimal under their own constraints will then be introduced to compare with the SS benchmark. This comparison will give an example of how the SS benchmark is used and show what optimal input is best for different conditions. Finally, we will expound on the importance of benchmarking motion control inputs.

II. Dimensionless System Model

The system used in this paper is the standard two-mass system (shown in Fig. 1) with one flexible mode and one rigid-body mode. Damping is included to make the results as general as possible. The goal of the point-to-point motion control of this system is to move the endpoint mass m_2 a distance x_f in minimum time subject to actuator constraint $F_{\max}$. Lumping several of these parameters into one makes it easier to explore the time-optimal solution space:

Table 1 Definition of dimensionless parameters

Parameter	Definition
Dimensionless time	$\omega_n t$
Dimensionless position	$[(\omega_n T_s)^2/2\pi](X_2/x_f)$
Dimensionless acceleration (a is the acceleration of m_2)	$(4/2\pi)(m/F_{\max})a$
Dimensionless force	$F/F_{\max}$

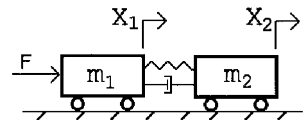

Fig. 1 Two-mass system with flexibility and damping.

$$(\omega_n T_s)^2 = \frac{4(m_1 + m_2)x_f\omega_n^2}{F_{\max}} \tag{1}$$

The parameter $\omega_n T_s$ represents the dimensionless move time of the SS input with T_s as the dimensional SS move time. Given actuator force constraint $F_{\max}$, no other input will move the rigid body mode a distance x_f in a shorter dimensionless time than $\omega_n T_s$.

The natural frequency ω_n is given by

$$\omega_n = \sqrt{(k/m_1)[1 + (m_1/m_2)]} \tag{2}$$

and the damping ratio ζ is given by

$$\zeta = \frac{b}{2}\sqrt{\frac{1 + (m_1/m_2)}{km_1}} \tag{3}$$

where k and b are the spring stiffness and the damping coefficient, respectively. The analysis was done with dimensionless parameters so that the results could easily be applied to other systems. Table 1 gives the definitions of the other dimensionless parameters used.

Because the acceleration of the unforced mass is of interest, we can find a second-order transfer function between the dimensionless acceleration a^* of mass m_2 and dimensionless force F^*:

$$\frac{A^*(s)}{F^*(s)} = \frac{4}{2\pi}\,\frac{2\zeta s + 1}{s^2 + 2\zeta s + 1} \tag{4}$$

where s is a dimensionless frequency that is normalized with ω_n. The factor of $4/2\pi$ comes from the particular choice of dimensionless acceleration, which is further explained in the next section.

A controllable canonical form state-space representation of this transfer function is given by

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}}_{A}\boldsymbol{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} F^*, \qquad y = a^* = \underbrace{\frac{4}{2\pi}[1 \quad 2\zeta]}_{C}\boldsymbol{x} \tag{5}$$

where $\boldsymbol{x}$ is the dimensionless state vector, y is the dimensionless acceleration of the unforced mass, and ζ is the damping ratio.

III. Move Plus Settling Time for the SS Benchmark

The general form for the response of a state-space system to an arbitrary input can be expressed as

$$\boldsymbol{x}(t^*) = e^{At^*}\boldsymbol{x}_0 + \int_0^{t^*} e^{A(t^* - \tau^*)}\boldsymbol{B}F^*(\tau^*)\,\mathrm{d}\tau^* \tag{6}$$

where t^* is dimensionless time, normalized by ω_n. If initial conditions are zero, the output acceleration can be written as

$$y(t^*) = \boldsymbol{CT}e^{\Lambda t^*}\left[\int_0^{t^*} e^{-\Lambda\tau^*}F^*(\tau^*)\,\mathrm{d}\tau^*\right]\boldsymbol{T}^{-1}\boldsymbol{B} \tag{7}$$

where $\boldsymbol{T}$ and Λ are given by

$$\boldsymbol{T} = \begin{bmatrix} 1 & 1 \\ -\zeta + j\sqrt{1-\zeta^2} & -\zeta - j\sqrt{1-\zeta^2} \end{bmatrix} \tag{8}$$

$$\Lambda = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \begin{bmatrix} -\zeta + j\sqrt{1-\zeta^2} & 0 \\ 0 & -\zeta - j\sqrt{1-\zeta^2} \end{bmatrix} \tag{9}$$

Equation (7) can be simplified to yield

$$y(t^*) = a^*(t^*) = \boldsymbol{CT}\frac{e^{\Lambda t^*}}{-2j\sqrt{1-\zeta^2}}\begin{bmatrix} -F^*(s_1) \\ F^*(s_2) \end{bmatrix} \tag{10}$$

where $s_1 = -\zeta + j\surd(1-\zeta^2)$ and $s_2 = -\zeta - j\surd(1-\zeta^2)$. This form is convenient because it neatly separates the dynamics of the system from the input. The only needed input parameter is its Laplace transform $[F^*(s_i)]$ evaluated at the complex dimensionless frequency s_1. To have no residual acceleration, this value must be zero.[6]

The dimensionless residual acceleration and jerk (directly after the input is turned off) are used to find the dimensionless peak acceleration (after the input is turned off) of a general system with an SS input. The peak acceleration is then assumed to decay at a rate described by $e^{-\zeta t^*} = e^{-\zeta\omega_n t}$. The settling time of the residual acceleration is found by defining a settling region. The total move plus settling time can then be compared to the move plus settling times from other motion control inputs.

Equation (10) can be expanded to give expressions for the residual acceleration a_0^* and jerk J_0^* when the input is turned off at $t^* = \omega_n T_s$,

$$y(\omega_n T_s) = a_0^* = \frac{4}{2\pi}\left[\frac{-\left[(1-2\zeta^2) + j2\zeta\sqrt{1-\zeta^2}\right]e^{s_1\omega_n T_s}F^*(s_1)}{-2j\sqrt{1-\zeta^2}} + \frac{\left[(1-2\zeta^2) - j2\zeta\sqrt{1-\zeta^2}\right]e^{s_2\omega_n T_s}F^*(s_2)}{-2j\sqrt{1-\zeta^2}}\right] \tag{11}$$

$$\dot{y}(\omega_n T_s) = J_0^* = \frac{4}{2\pi}\left[\frac{\left[(4\zeta^2-3)\zeta + j(1-4\zeta^2)\sqrt{1-\zeta^2}\right]e^{s_1\omega_n T_s}F^*(s_1)}{2j\sqrt{1-\zeta^2}} - \frac{\left[(4\zeta^2-3)\zeta - j(1-4\zeta^2)\sqrt{1-\zeta^2}\right]e^{s_2\omega_n T_s}F^*(s_2)}{2j\sqrt{1-\zeta^2}}\right] \tag{12}$$

The acceleration at dimensionless time t^* after the input is turned off is given by the free response as

$$a^*(t^*) = e^{-\zeta t^*}\left[A\cos\left(\sqrt{1-\zeta^2}t^*\right) + B\sin\left(\sqrt{1-\zeta^2}t^*\right)\right] \tag{13}$$

Because the maximum acceleration occurs at $\mathrm{d}a^*/\mathrm{d}t^* = 0$, the time when the peak acceleration occurs, t_p^*, can be expressed as

$$\tan\left(\sqrt{1-\zeta^2}\,t_p^*\right) = J_0^* \Big/ \left[\frac{\zeta}{\sqrt{1-\zeta^2}}\left(J_0^* + \zeta a_0^*\right) + \sqrt{1-\zeta^2}\,a_0^*\right] \tag{14}$$

Substituting Eq. (14) into Eq. (13) and simplifying gives

$$a^*\left(t_p^*\right) = e^{-\zeta t_p^*}\sqrt{\left(J_0^*\right)^2 + 2\zeta a_0^* J_0^* + \left(a_0^*\right)^2} \tag{15}$$

Unfortunately, t_p^* cannot be written in closed form. However, Eq. (15) can be approximated by assuming that the peak acceleration occurs when the input is turned off, where $t_p^* = 0$. This creates an upper bound on the peak acceleration, making the estimate slightly higher than the actual peak. The error caused by this approximation is small, less than 5% for a damping ratio under 0.6. The approximate peak acceleration is given by

$$a_p^* = a^*\left(t_p^*\right) \approx \sqrt{\left(J_0^*\right)^2 + 2\zeta a_0^* J_0^* + \left(a_0^*\right)^2} \tag{16}$$

Equations (11) and (12) can be substituted into Eq. (16) to solve for the peak residual acceleration as a function of ζ and input Laplace transform $F^*(s_1)$. After manipulation, the peak acceleration can be solved as

$$a_p^* = (4/2\pi)\tfrac{1}{2}\left(1\Big/\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n T_s}\left|F^*(s_1)\right| \times\sqrt{2\sqrt{\alpha^2+\beta^2}\cos\left(\theta + 2\omega_n T_s\sqrt{1-\zeta^2} + 2\phi_1\right) + 4\gamma} \tag{17}$$

where

$$\alpha = -64\zeta^8 + 80\zeta^6 - 16\zeta^4, \qquad \beta = \zeta\sqrt{1-\zeta^2}$$

$$\theta = \tan^{-1}(-\beta/\alpha), \qquad \phi_1 = \angle F^*(s_1)$$

$$\gamma = -32\zeta^7 + 40\zeta^5 - 8\zeta^3 - \zeta + 1$$

The quantities $|F^*(s_1)|$ and $\angle F^*(s_1)$ are found from the Laplace transform of the input. The Laplace transform of the input evaluated at the complex natural frequency $s_1 = -\zeta + j\surd(1-\zeta^2)$ can be expressed as

$$F^*(s_1) = \frac{\left[1 - 2e^{\zeta t_1^*}\cos\left(\sqrt{1-\zeta^2}\,t_1^*\right) + e^{\zeta\omega_n T_s}\cos\left(\sqrt{1-\zeta^2}\,\omega_n T_s\right)\right]}{-\zeta + j\sqrt{1-\zeta^2}} + \frac{j\left[2e^{\zeta t_1^*}\sin\left(\sqrt{1-\zeta^2}\,t_1^*\right) - e^{\zeta\omega_n T_s}\sin\left(\sqrt{1-\zeta^2}\,\omega_n T_s\right)\right]}{-\zeta + j\sqrt{1-\zeta^2}} \tag{18}$$

where t_1^* is the dimensionless time at which the input switches sign.

To find the settling time of the oscillation, an appropriate settling region must be defined. To keep the results as general as possible, a dimensionless allowable acceleration a_{allow}^* is used to define the settling region. This dimensionless parameter is defined by

$$a_{\text{allow}}^* = (4/2\pi)(m/F_{\max})\,a_{\text{allow}} \tag{19}$$

where m is the total mass $(m_1 + m_2)$, $F_{\max}$ is the peak force, and a_{allow} is the acceleration threshold. The factor $4/2\pi$ was chosen to normalize the peak acceleration to one for typical system parameters. This parameter permits the user to specify an allowable acceleration threshold and calculate a benchmark that accurately models that particular system. The larger the a_{allow}^* value, the shorter the settling time will be for the SS input. Figure 2 shows two different a_{allow}^* values and how they define the move plus settling time.

If the tolerance is expressed in terms of position error Δx, it can be converted into an a_{allow}^* value with the following formula:

$$a_{\text{allow}}^* = \left[(\omega_n T_s)^2\big/2\pi\right](\Delta x/x_f) \tag{20}$$

Because dimensionless acceleration amplitude will decay exponentially with initial value a_p^* and time constant $1/\zeta$, it will fall within the threshold specified by a_{allow}^* at the time defined as the move plus settling time. This creates a simple expression that can be used to find the move plus settling time. With the bounds dictated by a_{allow}^*, the dimensionless move plus settling time of an SS input can be expressed as

$$t_{\text{move+settle}}^* = \omega_n T_s + (1/\zeta)\ln\left(a_p^*\big/a_{\text{allow}}^*\right) \tag{21}$$

Equation (21) is the approximate closed-form solution for the SS benchmark's move plus settling time. If a particular motion control strategy has a move plus settling time that is longer than this simple form, it is highly likely that the extra time and effort spent in its development is wasted. The SS input works best for systems with moderate or heavy damping and larger allowable accelerations. More specific information about when the SS input is best will be given in the following section.

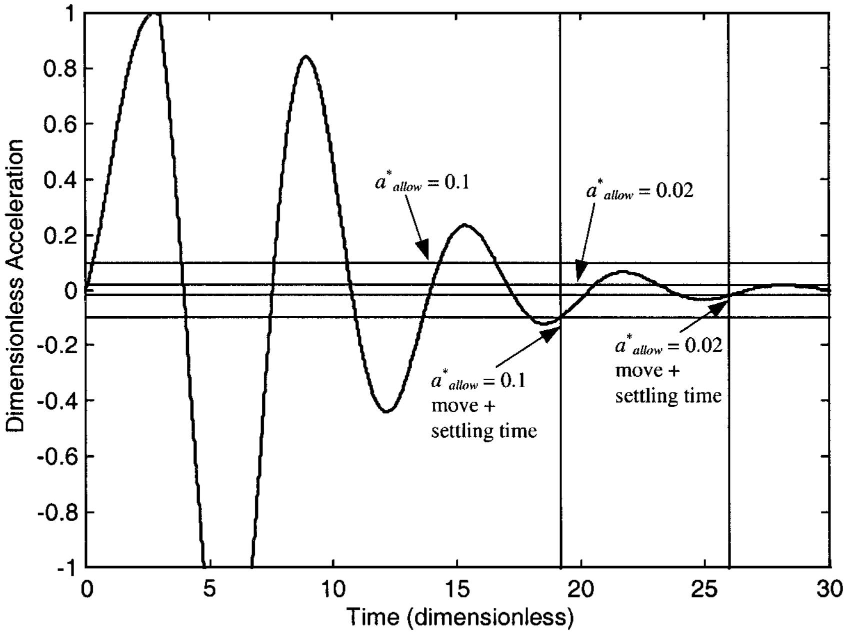

Fig. 2 Dimensionless acceleration response to an SS input: $\omega_n T_s = 6$ and $\zeta = 0.02$.

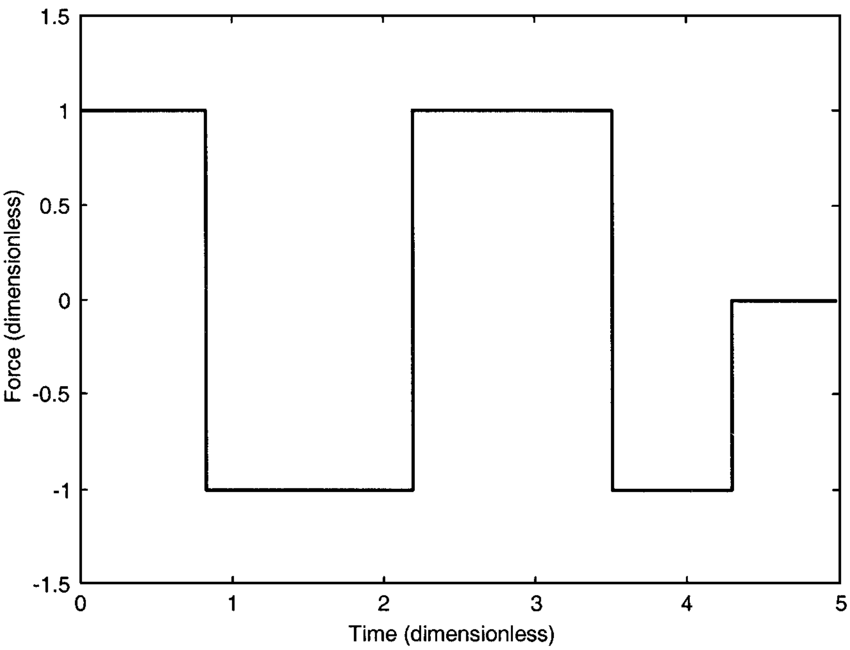

Fig. 3 TTO input for system with $\omega_n T_s = 2$ and $\zeta = 0.05$.

IV. Time-Optimal Inputs for Different Constraints

Whereas the SS solution gives the shortest possible move time for a rigid-body, subject to actuator constraints, the large amount of residual vibration associated with it makes it a poor choice for use on many real systems. The traditional time-optimal (TTO) input moves the rigid body the specified distance and cancels the flexible mode of the system. This cancellation comes from the addition of a complex pair of zeros that cancel the complex system poles.[13] The benefit of canceling the flexible mode is the elimination of residual vibration. The TTO input has two major drawbacks when compared to the SS input: 1) Move time increases with the added constraints. 2) The solution process is no longer closed form, but is numerically intensive.

Whereas the increase in move time is unavoidable, methods exist to efficiently solve for the traditional time-optimal solution. Normally, two different methods are used in time-optimal control problems: 1) constrained optimization[15] and 2) linear programming.[16] The constrained optimization method is efficient, but requires a good initial guess. The linear programming method finds the optimal solution without an initial guess, but is very inefficient. We found that combining these methods creates a process that is efficient and effective.[17] The first step is to generate a forcing function with a linear programming routine that has a large time step. Then use the transitions from positive to negative in the forcing function as initial guesses for switch times in a constrained optimization routine. Verifying the solution using a switching function[18] is the final step in the optimal solution process. Figure 3 shows a typical TTO input. Notice that, in general, it takes more than one switch to satisfy the additional constraints.

Tuttle[15] has found that by canceling the numerator dynamics of the system, faster maneuvers can be made. The cancellation is done by a pole that appears at the end of the command as a tail. This tail occurs after the move has taken place, and its length is not added into the move time because the desired position has been reached without residual vibration. This time-optimal with tail (TOT) input has the added constraint of zero cancellation, but produces move

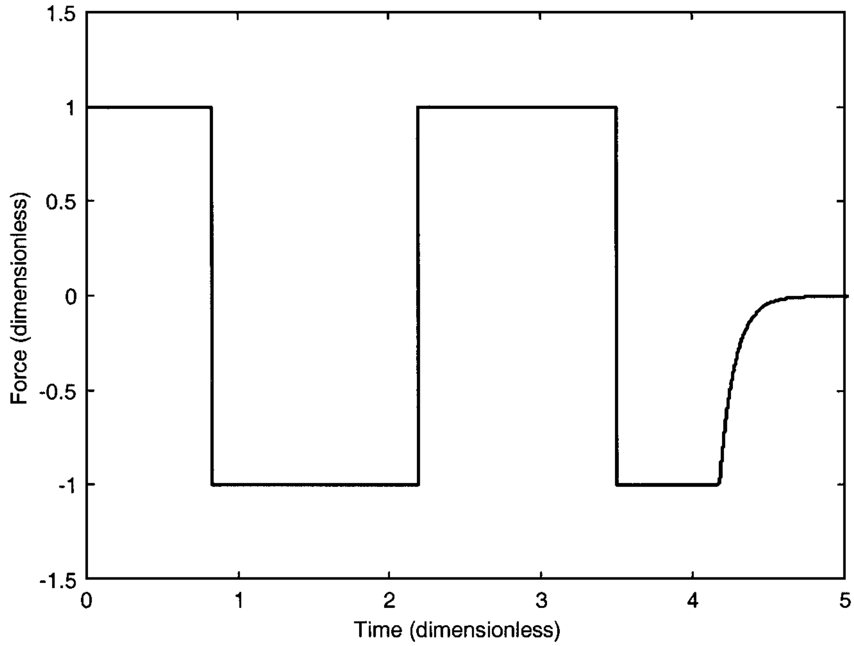

Fig. 4 TOT input for system with $\omega_n T_s = 2$ and $\zeta = 0.05$.

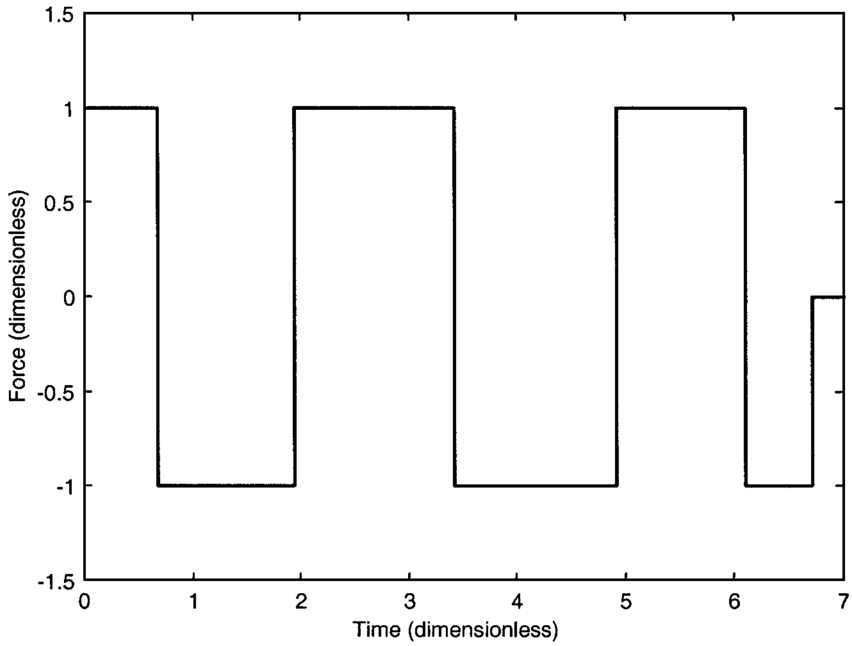

Fig. 5 ITTO input for system with $\omega_n T_s = 2$ and $\zeta = 0.05$.

times that are shorter than the TTO input in the presence of damping. For an undamped system, the inputs are the same because there is no zero to cancel. The same type of solution processes can be performed to solve for the TOT input, but its optimality cannot be verified like the TTO input.[15] Figure 4 shows a typical TOT input, clearly illustrating the tail beyond a dimensionless time of 4.2. Although the input must remain on during this time, the desired position is reached and maintained at the dimensionless time of 4.2.

A major criticism of time-optimal command shaping is its sensitivity to parameter uncertainty. Time-optimal inputs can be made less sensitive to frequency error by adding a constraint on the derivative of residual vibration with respect to frequency, which is equivalent to placing double zeros in the input for each system pole.[19] This input is referred to as the insensitive traditional time-optimal (ITTO) input. These additional constraints give three to four times more insensitivity, but with an additional time penalty. (The amount of insensitivity is defined by the range of frequency variation that keeps the residual vibration below a nominal value. A system with three to four times more insensitivity has a three to four times larger range of frequencies that keep the residual vibration below the same nominal value.) Figure 5 shows a typical ITTO input. Notice that, in general, it takes additional switches to satisfy the additional robustness constraints. The TOT could also be made less sensitive through the addition of double zeros. The results are very similar to the trends of the ITTO and so they are not included here. Table 2 gives a summary of the command shaping inputs compared in this paper.

The three inputs (TTO, TOT, and ITTO) are all time optimal under their own constraints, and they all represent excellent benchmarks for comparison to any motion control strategy. Unfortunately, they cannot be written in closed form like the SS input, and so the benchmarking process is a lot more complicated. The following section will compare all four inputs and explain how and why the SS input, and the other inputs when appropriate, should be used as benchmarks.

V. Benchmarking Move Plus Settling Time for Different Inputs

A benchmark is a tool for comparison and a standard to be measured against. The SS input is the simplest benchmark because its move plus settling time can be found in an approximate closed-form expression. However, in some cases it may not be the best benchmark. Figures 6 and 7 show the normalized move plus settling times of the SS, TTO, TOT, and ITTO inputs vs damping ratio for different move times $\omega_n T_s$. The move plus settling times have been normalized by dividing by $\omega_n T_s$. This is done to cancel the obvious effect that changing a parameter such as the move distance would have on the move plus settling time. The only difference between Figs. 6 and 7 is the value of $\omega_n T_s$. As mentioned before, $\omega_n T_s$ is a dimensionless variable that lumps several important point-to-point motion parameters. For a given application, this value can be determined from the total moving mass m, the peak available force $F_{\max}$, the desired travel distance x_f, and the natural frequency of oscillation ω_n.

As $\omega_n T_s$ is increased, the system becomes stiffer and more like a rigid body. For systems with $\omega_n T_s$ values greater than 40, the system becomes so rigid that there is little difference between the SS input and the other inputs. Because $\omega_n T_s/2\pi$ represents the number of natural periods in the command, systems with $\omega_n T_s$ values below 2π do not go through one cycle of motion during the move. This causes the output to have higher accelerations and, thus, makes the SS input look less favorable than in cases where the $\omega_n T_s$ value is moderate ($2\pi < \omega_n T_s < 40$). For undamped systems, when $\omega_n T_s/2\pi$ is an even integer, the vibration caused when the input is turned on is perfectly canceled by the vibration generated when the input reverses sign, achieving zero residual vibration. For damped systems, this perfect cancellation does not occur, but there is a significant reduction in residual vibration near these values. The $\omega_n T_s$ values chosen here stay away from this cancellation, ensuring the results are general. However, this further emphasizes the usefulness of considering the SS input just in case the system is near one of these special values of $\omega_n T_s$.

As damping is increased, the SS input looks more favorable in relation to other inputs. Figure 6 shows that the SS input with the larger settling region ($a^*_{\text{allow}} = 0.10$) has a lower move plus settling time than the ITTO when $\zeta > 0.3$. For heavily damped systems ($\zeta > 0.6$), the SS input creates a faster move than the TTO input. These trends are also seen in Fig. 7 for $\omega_n T_s = 10$, but only in the heavily damped cases is the SS input faster. Although these amounts of damping may seem far beyond most systems, a feedback controller can add significant closed-loop damping.

Note the move time advantage of the TOT input. As seen in both Figs. 6 and 7, it is the fastest of the four commands. Although it is up to 50% faster than the TTO input in heavily damped systems, the tail takes a long time to settle in these cases. Although the tail occurs after the system output has reached the desired position, it still needs to be applied to maintain equilibrium. In cases where repeated point-to-point motions need to be made in succession, the wait for the tail in the command profile to settle can make the overall move time much longer.

All of the SS results are highly dependent on the amount of allowable residual acceleration a^*_{allow}, which can be obtained from a position threshold Δx via Eq. (20). Clearly, larger allowable residual acceleration tolerance means shorter move plus settling time. Therefore, it is important that the motion control designer select an acceleration tolerance that is appropriate to the system. Only with a valid a^*_{allow} does the SS benchmark have any meaning.

Although the SS input can be a benchmark for any system, it is most appropriate for systems with damping ratios above 0.10. For lightly damped systems ($\zeta < 0.1$), the residual oscillation for the SS input takes much longer to settle than for the TTO or ITTO inputs. For these systems, the SS input should be an upper limit (in terms of move plus settling time) on all acceptable commands. Any inputs applied to lightly damped systems should perform better than the SS input, and the amount of improvement can serve as a performance index. Because the $\omega_n T_s$ value is the lower limit on pure move time, it can also be compared to the particular command. By knowing the lower bound, the motion control designer has a better feel for the time optimality of a particular command. For example, if an input has a move plus settling time that is very near the lower bound, further improvements in its design will bring diminishing results. For the conditions where the SS input is a poor input, the TTO or TOT inputs can be used as benchmarks.

Table 2 Summary of compared command shaping inputs

Input	End conditions
TTO	Rigid body achieves desired position and flexible mode vibration is eliminated at the end of the move.
TOT	Unforced mass achieves final position with zero final velocity at the specified final time but the forced mass continues moving. Tail that occurs after the final time serves to position the forced mass and absorb any energy left in the flexibility.
ITTO	Rigid body achieves desired position and flexible mode is cancelled. Additional zeros (switches) are added to reduce sensitivity to parameter variation.
SS	Rigid-body mode achieves desired position.

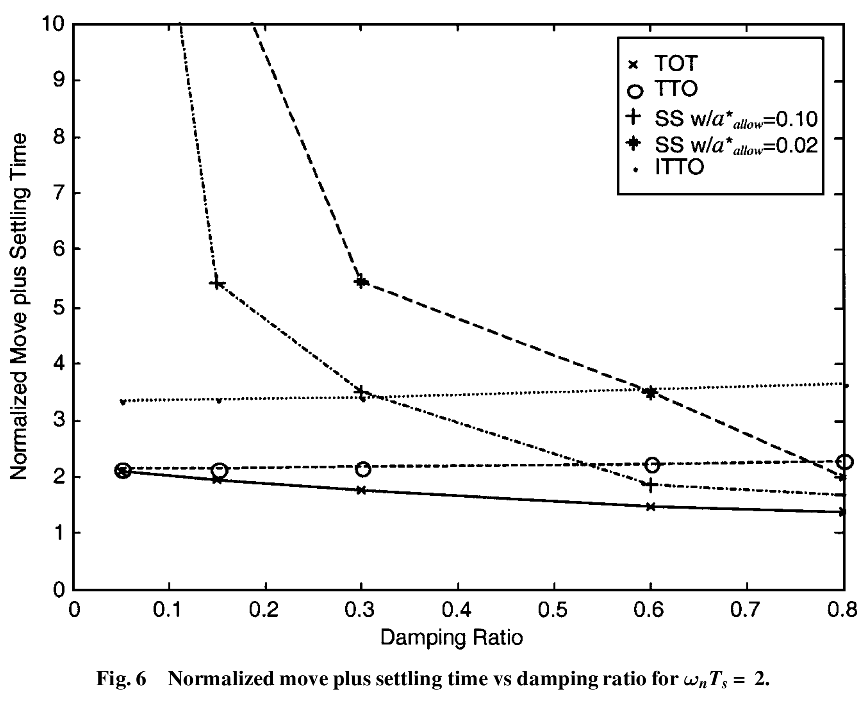

Fig. 6 Normalized move plus settling time vs damping ratio for $\omega_n T_s = 2$.

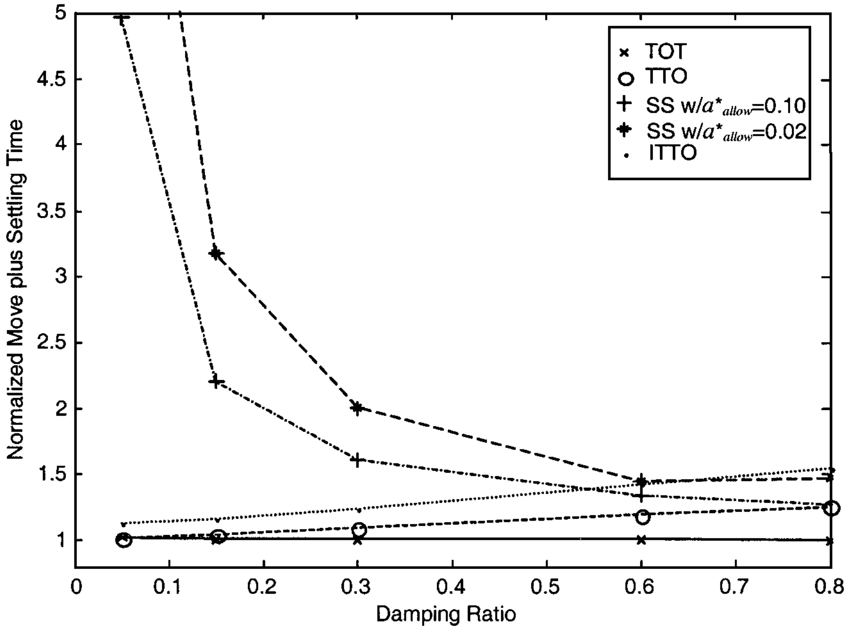

Fig. 7 Normalized move plus settling time vs damping ratio for $\omega_n T_s = 10$.

When insensitivity to parameter uncertainty is an issue, the SS input can be an excellent benchmark because it represents a highly insensitive input. When controller designs have move plus settling times that exceed those of the SS input, the design must be more insensitive to justify its longer move time. For lightly damped systems, the ITTO input would be a better benchmark. The ITTO input gives additional robustness with lightly damped systems at move times that are lower than those for the SS input. Although it is not a closed-form solution, the ITTO input is fairly easy to calculate using the solution process described here.

VI. Conclusions

Whereas a number of different benchmarks might be appropriate when comparing command shaping designs, it is imperative that some type of benchmarking be done. Researchers too often dismiss the time penalties associated with vibration cancellation or parameter insensitivity without comparing them to a standard. When a benchmark is applied, the case for a particular design is not only more general and better understood, but it is also stronger.

This paper has presented a useful approximate closed-form expression for determining the total move plus settling time for an SS input. This serves as the upper bound on move plus settling time for systems having actuator force limits and parameter variations. A lower bound is provided by the pure move time for an SS input. Thus, any other command shaping input can be compared against these limits to ensure that residual vibration reduction is effective and efficient. If a particular strategy has a move plus settling time that is longer than the move plus settling time for an SS input, it is very likely that the extra time and effort spent in its development is wasted. Several typical command profiles are compared against the SS benchmark to illustrate the effect of damping on move time. The results emphasize the importance of command shaping to minimize residual vibration when system damping ratio is less than 0.1.

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